Calculus I Review

Recall that the main objects in calculus are functions. A function is a rule of assignment. If we call a function $f(x)$, then f is going to be fed with a real number a and it will give another real number $f(a)$.

- The **domain** of a function f is the set of numbers that we can "plug in", for example, if $f(x) = \sqrt{x-1}$, we cannot set $x = 0$, so 0 is not in the domain of f. So the domain of f is $[1, \infty)$.
- The **range** of f is the possible values that $f(x)$ can take. In our example, the range is $[0, \infty)$.

One way to picture the functions is by means of its graph. For example:

- $f(x) = x^2$
- $g(x) = x^3$
- $h(x) = 1/x$
- $F(x) = \cos(x)$

Among all functions, in Calculus I some of them played special roles. A **continuous** function is function for which

$$\lim_{x \to a} f(x) = f(a)$$

for every a in its domain. A function is continuous if we can draw its graph without lifting the pencil. A **discontinuous** function might have jumps or blow up to infinity at a point. A continuous function is **differentiable** at a point a if

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists.}$$

The number $f'(a)$ represents the **slope** of the tangent line to $f(x)$ at $x = a$. If a functions is differentiable at every point in its domain, we get its derivative $f'(x)$. If the resulting function is also differentiable, we can take higher order derivatives. Recall that a differentiable function must be continuous, but a continuous function need not to be differentiable, like $f(x) = |x|$. Graphically:

- Discontinuous
- Continuous
- Differentiable
- Continuous, not differentiable

You should remember the basic differentiation formulas and the **chain rule**. For example, compute the derivative of $f(x) = \sqrt{\sin(2x)}$. Then $f'(x) = \frac{\cos(2x)}{\sqrt{\sin(2x)}}$. A function $F(x)$ is an **antiderivative** of $f(x)$ if $F'(x) = f(x)$. For example, $\sin x + \pi$ is an antiderivative of $\cos x$. In **integral** notation, this is written as:

$$\int \cos x \, dx = \sin x + C.$$

Graphically, a definite integral means the area under the curve $f(x)$ over the region of integration:

$$\int_0^\pi \cos x \, dx \text{ gives the area under } f(x) = \cos x \text{ from } 0 \text{ to } \pi.$$

In order to compute the value of a definite integral, we make use of the **Fundamental Theorem of Calulus**, which states that if g is a continuous function on $[a, b]$, then the function defined by

$$f(x) = \int_a^x g(t) \, dt$$

has derivative $f'(x) = g(x)$, for all $a < x < b$. One of the most important consequences of this theorem is that allows us to compute the integral of a function $h(x)$ over an interval $[a, b]$ if we know its antiderivative $H(x)$:

$$\int_a^b h(x) \, dx = H(b) - H(a).$$

Then,

$$\int_0^{\pi} \cos x \, dx = \sin(x)|_0^{\pi} = \sin(\pi) - \sin(0) = 0.$$

From this, it is not surprising that a big effort is going to be directed to computing antiderivatives of functions. We will do this in a systematic and somehow mechanic way. One example of this is the **substitution rule**, which **will play an essential role in this course**:

- Calculate $\int x(x^2+5)^{10} \, dx$.

- Calculate $\int \tan x \sec^2 x \, dx$.

- Calculate $\int x \cos(x^2) \, dx$.

More on Integrals

Theorem 1 *If f is continuous on $[a,b]$ or if f has a finite number of jump discontinuities, then f is integrable on $[a,b]$, i.e. $\int_a^b f(x)dx$ exists.*

Properties of integrals:

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x) = 0$

3. $\int_a^b c dx = c(b-a)$, where c is any constant.

4. $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx$

5. $\int_a^b c f(x) dx = c \int_a^b f(x) dx$, where c is any constant.

6. $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx$

7. $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx$

Remark: It is **not** true that $\int_a^b f(x)g(x)dx = \int_a^b f(x)dx \int_a^b g(x)dx$.

Comparison Properties:

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \geq 0$.

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$.

3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$